Novel analytic representations for caps, floors and collars on continuous flows, arbitrage-free relations and optimal investments*

José Carlos Dias†

Iscte - Instituto Universitário de Lisboa, Lisbon, Portugal.

Business Research Unit (BRU-IUL), Lisbon, Portugal.

João Pedro Vidal Nunes

Iscte - Instituto Universitário de Lisboa, Lisbon, Portugal.

Business Research Unit (BRU-IUL), Lisbon, Portugal.

Fernando Correia da Silva

Iscte - Instituto Universitário de Lisboa, Lisbon, Portugal.

*This work was supported by the Fundação para a Ciência e a Tecnologia [grant number UIDB/00315/2020].

†Corresponding author. Iscte - Instituto Universitário de Lisboa, Edifício II, Av. Prof. Aníbal Bettencourt, 1600-189 Lisboa, Portugal. Email: jose.carlos.dias@iscte.pt. Tel: +351 21 7650531

Novel analytic representations for caps, floors and collars on continuous flows, arbitrage-free relations and optimal investments

Abstract

The valuation of finite maturity caps, floors and collars on continuous flows has been made using perpetual methods inspired in the real options literature. In particular, the value of such finite-lived contractual arrangements is obtained by subtracting the risk-neutral expectation of the forward starting perpetual solution from the corresponding (current) perpetual solution. In this paper, we produce formulae for finite maturity profit caps and profit floors that are contingent on continuous flows without the need of using such time decomposition technique. By doing so, we are able to directly tackle the valuation of finite-lived profit caps and profit floors. The related price caps, price floors and price collars are shown to be easily obtained from any analytic representation of profit caps and profit floors using some arbitrage-free relations. We also show how to derive the Greeks of all these finite-lived contractual arrangements. Finally, we offer two novel methods that allow us to calculate the optimal triggers of investment projects in the presence of price floors and price collars regimes in a way that is much simpler than the ones currently used in the literature.

Keywords: Caps, floors and collars; Continuous flows; Arbitrage-free; Investment opportunities.

1. Introduction

Caps and floors are contractual agreements that are commonly used in interest rate and commodity products aiming to provide upper and lower bounds to the possible ranges of outcomes. These contracts are usually used in a multi-period setting, so that each element of a cap or a floor—known as caplet and floorlet, respectively—provides limits to the payoff values. In this context, a cap or floor is the sum of a series of caplets or floorlets (with each caplet or floolet operating over three or six months). Several applications of these contracts can be found, for example, in Black (1976), Boyle and Turnbull (1989) and Hull (2018).

The real options literature adapted the rational of caps and floors by showing that the cash flows from a project can also be viewed as a set of options on continuous flows. In particular, McDonald and Siegel (1985) argue that the functional form $J = \int_0^T v_0(t)\,dt$ can be understood as the value of an investment project producing a continuous cash flow $v_0(t)$ that captures the positive part of a stochastic net profit on time t while avoiding losses. Nevertheless, McDonald and Siegel (1985) focus their discussion on the derivation and properties of each caplet $v_0(t)$ and do not present the analytical solution for the functional form J.

Shackleton and Wojakowski (2007) provide analytic formulae for valuing finite maturity (profit) caps and floors that are contingent on continuous flows following a lognormal distribution. Therefore, they offer an efficient closed-form solution for determining the value of the finite-lived investment project of McDonald and Siegel (1985). To accomplish this purpose, they use perpetual methods inspired in the real options literature and evaluate the finite-lived case using the risk-neutral expectation of a forward start perpetuity. More specifically, and considering for instance the case of a finite-lived profit cap expiring in T years, its value is decomposed (or replicated) by a portfolio that includes a long position in a perpetual profit cap and a short position in a forward start perpetual profit cap that begins after T years. The use of this replicating portfolio is feasible because the individual caplets contained within each integral are independent. This time decomposition technique has emerged in the literature as the stepping-stone to many other recent applications of real options models with related caps, floors and collars—e.g., Barbosa et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022) and Paxson et al. (2022).

In a nutshell, we offer three main contributions to the literature. First, we produce novel analytic representations for valuing finite-lived profit caps and profit floors that are contingent on continuous flows, but without the need of using the replicating portfolio approach offered by Shackleton and Wojakowski (2007), i.e., that do not require the computation of forward starting perpetuities. By doing so, we are able to directly tackle the valuation of finite-lived profit caps and profit floors and, hence, provide a straight closed-form solution of the integral that is required for determining the value of the finite-lived investment project of McDonald and Siegel (1985). Moreover, we show that the usual put-call duality in the spirit of McDonald and Schroder (1998) is still valid for a sum of finite or infinite continuum of European-style caplets and floorlets, which implies that any finite maturity (resp., perpetual) profit cap solution can be used to evaluate the corresponding finite-lived (resp., perpetual) profit floor (or vice versa), with a simple and appropriate switching of model parameters.

Since Shackleton and Wojakowski (2007) have already produced analytical formulae for valuing finite maturity profit caps and profit floors, our first contribution might be seen as just an alternative way for evaluating such contingent claims. Even though this is true, we show, to the best of our knowledge for the first time, how the corresponding integrals with respect to time can be directly tackled analytically. Our novel equivalent analytical representation is feasible under the lognormal diffusion because it is possible to solve the integrals with respect to time using integration by parts, since in this case the problem involves only the cumulative density function of the standard univariate normal distribution.

However, the time decomposition approach of Shackleton and Wojakowski (2007) is very general because the replicating portfolio is valid regardless of the underlying stochastic process of the continuous flows. For example, the constant elasticity of variance (hereafter, CEV) diffusion considered in Dias and Nunes (2011) and Dias et al. (2024) involves noncentral chi-square distributions laws for which integrals with respect to time using integration by parts cannot be solved analytically. Fortunately, it is still possible to apply the replicating portfolio technique of Shackleton and Wojakowski (2007) because the risk-neutral expectations of forward start perpetuities under the CEV process can be calculated. Nevertheless, our analytical formulae might be considered, at least, an equivalent and viable alternative methodology for the case where the underlying process is assumed to follow a geometric Brownian motion, which is actually the most common assumption that is used in the vast majority of real options applications.

We recall that different, but related, contractual price arrangements with caps, floors and collars clauses are often used by policy makers as incentive vehicles for promoting investment decisions of private firms—see, for example, Barbosa et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022), Paxson et al. (2022) and the references contained therein. For instance, a price floor is a contract in which there is a floor level (or a minimum price guarantee) that protects the investor against the downside risk of adverse cash flows. By contrast, a price cap arrangement includes a cap level (or a maximum price guarantee) aiming that the investor foregoes abnormally favorable cash flows. A price collar arrangement contains both cap and floor clauses.

Even though price caps, price floors and price collars can be evaluated analytically using also a replicating portfolio approach, their direct link with the profit caps and profits floors studied in Shackleton and Wojakowski (2007) has not been properly acknowledged in the literature. To fill this gap, and as our second contribution, we show that such price caps, price floors and price collars are easily obtained from any analytic representation of profit caps and profit floors—e.g., the closed-form solution offered in Shackleton and Wojakowski (2007) or the alternative analytical methodology that is proposed in this paper—using some arbitrage-free relations. If any of these relationships is not obeyed, then an arbitrage opportunity emerges instantaneously. Moreover, we also show how to derive the Greeks of all these finite-lived contractual agreements.

As our last contribution, we offer two new methods that allow us to calculate the optimal triggers of investment in a way that is much simpler than the ones offered in Barbosa et al. (2018), Barbosa et al. (2020) and Barbosa et al. (2022)—for the case of projects with floor-style clauses—and Adkins et al. (2019), Barbosa et al. (2020) and Paxson et al. (2022)—for the case of projects with collar-style arrangements. The first method is based on the novel analytical representation proposed in this paper, whereas the second method uses the original formulae of finite-lived caps offered by Shackleton and Wojakowski (2007). Independently of the analytical representation that is chosen, we show that the computation complexity of the thresholds of investment is reduced to solving only a single nonlinear equation instead of two or three nonlinear equations that are presently required in the literature in the case of floor-style and collar-style contracts, respectively. The advantage of our two methods is its simplicity, especially when used in general-purpose computer programs such as spreadsheets.

The remainder of the paper is organized in the following way. Section 2 produces the novel analytic representations for valuing profit caps and profit floors and highlights the possibility of applying profit cap - profit floor dualities on continuous flows. Section 3 shows how to evaluate price caps and price floors using some arbitrage-free relations with profit caps and profit floors. Section 4 extends these arbitrage-free relations for valuing price collars arrangements. The derivation of Greeks of all these finite-lived contractual agreements is shown in Section 5. Section 6 provides two simple approaches for calculating the optimal thresholds of investment projects with price floors and price collars regimes. Finally, Section 7 provides the main conclusions. All the proofs are relegated to an appendix.

2. Profit caps and profit floors

The main goal of this section is to produce novel analytical formulae to evaluate (profit) caps and floors that are contingent on continuous flows following a lognormal distribution. Similarly to Shackleton and Wojakowski (2007), we consider a continuous price or revenue process $\{P_t \in \mathbb{R}^+ : t \geq 0\}$ that generates cash at an instantaneous rate of flow $P_t dt$ and that is governed by the risk-neutral dynamics

$$dP_t = (r - q) P_t dt + \sigma P_t dW_t^{\mathbb{Q}}, \tag{1}$$

where r, q and σ are the (positive and constant) risk free interest rate, dividend yield (or rate of return shortfall) and volatility, respectively, while $\{W_t^{\mathbb{Q}} \in \mathbb{R} : t \geq 0\}$ is a standard

Brownian motion under the risk-neutral measure $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.

Let $V(P_t, X, T)$ be the time-t value of a finite-lived profit cap. Its value can be derived from a cash flow of an instantaneous maximum flow rate $\Pi(P_t)\,dt$, with $\Pi(P_t) = (P_t - X)^+$, over stochastic revenue P_t and fixed cost X. Under the risk-neutral measure, its (current) time-0 value is given by

$$\begin{aligned} V(P_0, X, T) &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[\Pi(P_t)\right] dt \\ &= \int_0^T c_0(P_0, X, t)\,dt, \end{aligned} \tag{2}$$

where $c_0(P_0, X, t)$ is interpreted as the time-0 price of the European-style call option (or caplet) offered in Black and Scholes (1973) and Merton (1973) on an underlying asset with spot price P_0 (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed cost rate) and with expiry date at time t (≥ 0).

Similarly, the value of a finite-lived floor can be obtained as

$$\begin{aligned} F(P_0, X, T) &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[\Pi(P_t)\right] dt \\ &= \int_0^T p_0(P_0, X, t)\,dt, \end{aligned} \tag{3}$$

but now with $\Pi(P_t) = (X - P_t)^+$, while $p_0(P_0, X, t)$ is interpreted as the time-0 price of a European-style put option (or floorlet) given in Black and Scholes (1973) and Merton (1973) on an underlying asset with spot price P_0 (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed proceeds rate) and with expiry date at time t (≥ 0).

Next proposition produces novel formulae for valuing the contingent claims on continuous flows (2) and (3) that do not require the use of the time decomposition technique considered in Shackleton and Wojakowski (2007). This is accomplished by simply using integration by parts and, hence, it has the advantage that no risk-neutral expectations of forward start perpetuities are required.

In summary, the proposed novel approach allows us to directly tackle the finite maturity case—which is the most important one in the context of executive management decisions—

without the need of using the respective perpetual solutions. Nevertheless, the corresponding perpetual cap and floor claims, denoted by $V(P_0, X, \infty)$ and $F(P_0, X, \infty)$, respectively, arise immediately by taking $T \to \infty$.

Proposition 1 *Assume the lognormal process* (1) *with positive r, q and σ.*
*(i) The time-*0 *value of a finite-lived cap, $V(P_0, X, T)$, is given by*

$$V(P_0, X, T) = -\frac{P_0}{q} I(0, T, P_0, X, 1, q) + \frac{X}{r} I(0, T, P_0, X, -1, r), \tag{4}$$

where

$$\begin{aligned} & I(0, T, P_0, X, \phi, v) \\ = \; & e^{-vT} N\left(\frac{a}{\sqrt{T}} + b_\phi \sqrt{T}\right) - N(d(0)) \\ & -\frac{1}{2}\left(\frac{b_\phi}{c_v} + 1\right) e^{a\left(c_v - b_\phi\right)} \left[N\left(\frac{a}{\sqrt{T}} + c_v\sqrt{T}\right) - N(d(0))\right] \\ & -\frac{1}{2}\left(\frac{b_\phi}{c_v} - 1\right) e^{-a\left(c_v + b_\phi\right)} \left[N\left(-\frac{a}{\sqrt{T}} + c_v\sqrt{T}\right) - 1 + N(d(0))\right], \end{aligned} \tag{5}$$

with $N(.)$ representing the cumulative density function of the standard univariate normal distribution,

$$a := \frac{\ln(P_0/X)}{\sigma}, \tag{6}$$

$$b_\phi := \frac{r - q + \phi\sigma^2/2}{\sigma}, \tag{7}$$

$$c_v := \sqrt{b_\phi^2 + 2v}, \tag{8}$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}^+$, and[1]

$$N(d(0)) := 1 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 0 \times \mathbb{1}_{\{P_0 < X\}}. \tag{9}$$

*(ii) The time-*0 *value of a finite-lived floor, $F(P_0, X, T)$, is given by*

$$F(P_0, X, T) = V(P_0, X, T) + \frac{X}{r}\left(1 - e^{-rT}\right) - \frac{P_0}{q}\left(1 - e^{-qT}\right). \tag{10}$$

[1] Even though the parameter defined in equation (8) could be formally stated as $c_{\phi,v}$, we only define it by c_v to lighten notation and because c_q and c_r are always associated with $\phi = 1$ and $\phi = -1$, respectively.

*(iii) The time-*0 *value of a perpetual cap,* $V(P_0, X, \infty)$*, is given by*

$$V(P_0, X, \infty) = -\frac{P_0}{q} I(0, \infty, P_0, X, 1, q) + \frac{X}{r} I(0, \infty, P_0, X, -1, r), \tag{11}$$

where

$$\begin{aligned} I(0, \infty, P_0, X, \phi, v) \quad &= \quad -N(d(0)) - \frac{1}{2}\left(\frac{b_\phi}{c_v} + 1\right) e^{a\left(c_v - b_\phi\right)} \left[1 - N(d(0))\right] \\ &\quad -\frac{1}{2}\left(\frac{b_\phi}{c_v} - 1\right) e^{-a\left(c_v + b_\phi\right)} N(d(0)). \end{aligned} \tag{12}$$

*(iv) The time-*0 *value of a perpetual floor,* $F(P_0, X, \infty)$*, is given by*

$$F(P_0, X, \infty) = V(P_0, X, \infty) + \frac{X}{r} - \frac{P_0}{q}. \tag{13}$$

Proof. Please see Appendix A. ■

Remark 1 *Clearly, the analytical solutions for the perpetual cap* (11) *and the finite-lived cap* (4) *can be immediately used to analyze the costless dynamic entry and exit investment problems considered in Keswani and Shackleton (2006, Section 6).*

Although the closed-form solutions (10) and (13) have been derived without explicitly invoking the caplet-floorlet parity, such arbitrage-free relation is, as expected, observed in both cases. Another interesting arbitrage-free relation for profit caps and profit floors on continuous flows can be also stated using the insights of McDonald and Schroder (1998). In particular, they show that a put-call duality holds whenever the price of a put option can be recovered from the price of a call option (and vice versa) through a suitable change in its arguments—more specifically, by changing the spot price by the strike and the interest rate by the dividend yield.

Taking the plain-vanilla call (or caplet) and put (or floorlet) solutions offered in Black and Scholes (1973) and Merton (1973) explicitly defined as functions of all of their input parameters, it follows that

$$c_0(P_0, X, T, \sigma, r, q) = p_0(X, P_0, T, \sigma, q, r). \tag{14}$$

Since this is true for any expiry date T, then it should be valid also for a sum of finite or infinite continuum of European-style caplets and floorlets. Therefore, the following two profit cap-profit floor dualities on continuous flows will be also true:

$$V(P_0, X, T, \sigma, r, q) = F(X, P_0, T, \sigma, q, r) \tag{15}$$

and

$$V(P_0, X, \infty, \sigma, r, q) = F(X, P_0, \infty, \sigma, q, r). \tag{16}$$

The intuition for equation (14) rests on the duality between the underlying asset and cash. A call option gives the right (but not the obligation) to exchange cash for the asset, while a put option gives the right (but not the obligation) to exchange the asset for cash. This duality result implies that any finite maturity or perpetual profit cap solution can be used to evaluate the corresponding finite-lived or perpetual profit floor (or the other way around) with a simple substitution of parameters, as shown in equations (15) and (16).

3. Price caps and price floors

The goal now is to analyze the price caps and price floors in the spirit of Barbosa et al. (2018), Adkins and Paxson (2019), Barbosa et al. (2020) and Barbosa et al. (2022) that are expressed also in the form of perpetual real options and, therefore, require the use of the time decomposition technique considered in Shackleton and Wojakowski (2007). Using the novel results of Proposition 1, we show that the finite-lived price caps and price floors can be evaluated without the need of computing risk-neutral expectations of forward start perpetuities. Again, the corresponding perpetual claims arise immediately by taking $T \to \infty$ in the finite maturity solutions.

3.1. Price caps

A price cap provides an instantaneous flow rate $\Pi(P_t) = \min(P_t, H)$ and can be thought as the value of an active project containing a cap level H that provides a ceiling to a given market price. Considering for instance a project associated to a given energy generating facility, the ceiling level H prevents governments to pay excessive rents to producers of energy and shields final customers against escalating prices.

Noting that $\min(P_t, H) = H - \max(H - P_t, 0)$, it is straightforward to conclude that the finite-lived price cap can be represented by

$$\begin{aligned} V_c(P_0, H, T) &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[H - (H - P_t)^+\right] dt \\ &= H\int_0^T e^{-rt}dt - \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(H - P_t)^+\right] dt \\ &= \frac{H}{r}\left(1 - e^{-rT}\right) - F(P_0, H, T), \end{aligned} \tag{17}$$

with $F(P_0, H, T)$ being the finite maturity profit floor (10) with a strike $X = H$. This originates an arbitrage-free relation between a finite-lived price cap and a finite maturity profit floor.

Alternatively, since $\min(P_t, H) = P_t - \max(P_t - H, 0)$, it is possible to show that the finite-lived price cap can be also calculated as

$$\begin{aligned} V_c(P_0, H, T) &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[P_t - (P_t - H)^+\right] dt \\ &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[P_t\right] dt - \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(P_t - H)^+\right] dt \\ &= \frac{P_0}{q}\left(1 - e^{-qT}\right) - V(P_0, H, T), \end{aligned} \tag{18}$$

with $V(P_0, H, T)$ being the finite maturity profit cap (4) with a strike $X = H$. In this case, this originates an arbitrage-free relation between a finite-lived price cap and a finite maturity profit cap.

The corresponding perpetual price cap can be obtained either via a perpetual profit floor or through a perpetual profit cap by taking the limit $T \to \infty$ of the analytic representations (17) and (18), respectively, thus resulting in

$$\begin{aligned} V_c(P_0, H, \infty) &= \lim_{T\to\infty} V_c(P_0, H, T) \\ &= \frac{H}{r} - F(P_0, H, \infty) && (19) \\ &= \frac{P_0}{q} - V(P_0, H, \infty). && (20) \end{aligned}$$

Clearly, equating the solutions (17) and (18) yields the finite-lived caplet-floorlet parity relation (10), whereas the combination of the perpetual solutions (19) and (20) obeys the infinite caplet-floorlet parity relation (13).

Remark 2 *The arbitrage-free relations* (19) *and* (20) *constitute viable alternatives for computing the value of an active project of a firm in the spirit of Adkins and Paxson (2019, equation 14).*

3.2. Price floors

A price floor generates an instantaneous flow rate $\Pi(P_t) = \max(P_t, L)$ and can be understood as the value of an active project containing a floor level L that guarantees a minimum to the prevailing market price in the face of adverse circumstances. Considering for example a project associated to a given energy generating facility, the floor level L aims to prevent the downside risk to producers of energy.

Noting that $\max(P_t, L) = \max(P_t - L, 0) + L$, it is straightforward to conclude that the finite-lived price floor can be represented by

$$\begin{aligned} V_f(P_0, L, T) &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(P_t - L)^+ + L\right] dt \\ &= \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(P_t - L)^+\right] dt + L\int_0^T e^{-rt} dt \\ &= V(P_0, L, T) + \frac{L}{r}\left(1 - e^{-rT}\right), \end{aligned} \tag{21}$$

with $V(P_0, L, T)$ being the finite maturity profit cap (4) with a strike $X = L$. This originates an arbitrage-free relation between a finite-lived price floor and a finite maturity profit cap. In particular, definition (21) will be used for calculating the optimal thresholds of investment in the presence of a price floor regime that is covered in Section 6.

Alternatively, since $\max(P_t, L) = \max(L - P_t, 0) + P_t$, it is possible to show that the finite-lived price floor can be also computed as

$$\begin{aligned} V_f(P_0, L, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[(L - P_t)^+ + P_t \right] dt \\ &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[(L - P_t)^+ \right] dt + \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} \left[P_t \right] dt \\ &= F(P_0, L, T) + \frac{P_0}{q} \left(1 - e^{-qT} \right), \end{aligned} \tag{22}$$

with $F(P_0, L, T)$ being the finite maturity profit floor (10) with a strike $X = L$. In this case, this originates an arbitrage-free relation between a finite-lived price floor and a finite maturity profit floor.

Remark 3 *The arbitrage-free relations* (21) *and* (22) *can be understood as viable alternatives for evaluating the value of an active project of a firm in the spirit of Barbosa et al. (2018, equation 19).*

The corresponding perpetual price floor can be obtained either via a perpetual profit cap or through a perpetual profit floor by taking the limit $T \to \infty$ of the analytic representations (21) and (22), respectively, thus yielding

$$\begin{aligned} V_f(P_0, L, \infty) &= \lim_{T \to \infty} V_f(P_0, L, T) \\ &= V(P_0, L, \infty) + \frac{L}{r} \qquad (23) \\ &= F(P_0, L, \infty) + \frac{P_0}{q}. \qquad (24) \end{aligned}$$

Clearly, equalizing the solutions (21) and (22) yields the finite-lived caplet-floorlet parity relation (10), whereas the combination of the perpetual solutions (23) and (24) obeys the infinite caplet-floorlet parity relation (13).

Remark 4 *The arbitrage-free relations* (23) *and* (24) *can be understood as viable alternatives for evaluating the value of an active project of a firm in the spirit of Barbosa et al. (2018, equation 8) and Adkins and Paxson (2019, equation 12).*

3.3. Price caps, price floors and vertical bullish spreads

A portfolio composed by a long price cap and a long price floor can be defined in terms of a vertical bullish spread. Combining equations (18) and (21) as well as equations (17) and (22) yields, respectively,

$$
\begin{aligned}
& V_c(P_0, H, T) + V_f(P_0, L, T) \\
= \; & \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{P_0}{q}\left(1 - e^{-qT}\right) + V(P_0, L, T) - V(P_0, H, T) \qquad (25) \\
= \; & \frac{H}{r}\left(1 - e^{-rT}\right) + \frac{P_0}{q}\left(1 - e^{-qT}\right) + F(P_0, L, T) - F(P_0, H, T), \qquad (26)
\end{aligned}
$$

where $V(P_0, L, T) - V(P_0, H, T)$ is interpreted as a bull profit cap spread (i.e., a long profit cap with lower strike L and a short profit cap with higher strike H), whereas $F(P_0, L, T) - F(P_0, H, T)$ is understood as a bull profit floor spread (i.e., a long profit floor with lower strike L and a short profit floor with higher strike H).

The special case with $L = H := X$ results in

$$
V_c(P_0, X, T) + V_f(P_0, X, T) = \frac{X}{r}\left(1 - e^{-rT}\right) + \frac{P_0}{q}\left(1 - e^{-qT}\right), \qquad (27)
$$

which corresponds to a portfolio (without restrictions in the flow P_t) paying a cash flow rate $(X + P_t)\, dt$. Again, the relations (25), (26) and (27) must be satisfied in order to avoid the existence of arbitrage opportunities.

4. Price collars

Let us now consider the price collars studied in Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020) and Paxson et al. (2022), which have been represented in the form of perpetual methods valued through the portfolio decomposition approach of Shackleton and Wojakowski (2007).

In a price collar arrangement the price floats freely subject to a price floor level L and a price cap level H (with $H \geq L$), so that the investor receives L if $P_t < L$, receives the unit price P_t if $L \leq P_t < H$ and receives H if $P_t \geq H$. Its instantaneous flow rate is defined as $\Pi(P_t) = \min(\max(L, P_t), H)$, that is

$$\Pi(P_t) = L\mathbb{1}_{\{P_t<L\}} + P_t\mathbb{1}_{\{L\leq P_t<H\}} + H\mathbb{1}_{\{P_t\geq H\}}. \tag{28}$$

Clearly, this payoff structure can be analyzed under different perspectives. For instance, the payoff (28) can be understood as a portfolio of path-independent binary options in the sense of Rubinstein and Reiner (1991) with the following components: (i) a European-style cash-or-nothing put on continuous flows with a strike price equal to the predetermined fixed cash amount L; (ii) a European-style range asset on the flow rate P_t, with lower strike L and upper strike H; and (iii) a European-style cash-or-nothing call on continuous flows with a strike price equal to the predetermined fixed cash amount H.

Apart from this interpretation, the payoff (28) can also be expressed in terms of profit caps, profit floors, price caps or price floors. Such arbitrage-free relations will be presented next.

4.1. Finite-lived price collars expressed in terms of profit caps

It is straightforward to show that the payoff (28) is equivalent to

$$\Pi(P_t) = L + (P_t - L)\mathbb{1}_{\{P_t\geq L\}} - (P_t - H)\mathbb{1}_{\{P_t\geq H\}}. \tag{29}$$

Therefore, a finite-lived price collar can be expressed as

$$\begin{aligned} C(P_0, L, H, T) &= L\int_0^T e^{-rt}dt + \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(P_t - L)^+\right]dt - \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(P_t - H)^+\right]dt \\ &= \frac{L}{r}\left(1 - e^{-rT}\right) + V(P_0, L, T) - V(P_0, H, T) \qquad (30) \\ &= V_f(P_0, L, T) - V(P_0, H, T), \qquad (31) \end{aligned}$$

with $V(P_0, X, T)$, for $X \in \{L, H\}$, and $V_f(P_0, L, T)$ being the finite maturity profit cap (4) and the finite maturity price floor (21), respectively.

This implies that the finite-lived price collar (30) can be interpreted as a portfolio composed by: (i) a cash amount with a present value equal to $\frac{L}{r}\left(1 - e^{-rT}\right)$ that can be understood as the sum of long forward contracts with instantaneous delivery price L; (ii) a long position in a finite-lived profit cap with fixed cost L; and (iii) a short position in a finite-lived profit cap with fixed cost H. In particular, definition (30) will be used for determining the optimal triggers of investment in the presence of a price collar regime that is covered in Section 6. Moreover, the equivalent finite-lived price collar (31) can be understood as a portfolio composed by: (i) a long position in a finite-lived price floor with a fixed guarantee L; and (ii) a short position in a finite-lived profit cap with fixed cost H.

4.2. Finite-lived price collars expressed in terms of profit floors

It can be shown that the payoff (28) is equivalent to

$$\Pi(P_t) = H + (L - P_t)\mathbb{1}_{\{P_t<L\}} - (H - P_t)\mathbb{1}_{\{P_t<H\}}. \tag{32}$$

Therefore, a finite-lived price collar can be expressed as

$$\begin{aligned} C(P_0, L, H, T) &= H\int_0^T e^{-rt}dt + \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(L - P_t)^+\right]dt - \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(H - P_t)^+\right]dt \\ &= \frac{H}{r}\left(1 - e^{-rT}\right) + F(P_0, L, T) - F(P_0, H, T) \qquad (33) \\ &= V_c(P_0, H, T) + F(P_0, L, T), \qquad (34) \end{aligned}$$

with $F(P_0, X, T)$, for $X \in \{L, H\}$, and $V_c(P_0, H, T)$ being the finite maturity profit floor (10) and the finite maturity price cap (17), respectively.

This implies that the finite-lived price collar (33) can be interpreted as a portfolio composed by: (i) a cash amount with a present value equal to $\frac{H}{r}\left(1 - e^{-rT}\right)$ that can be understood as the sum of long forward contracts with instantaneous delivery price H; (ii) a long position in a finite-lived profit floor with a fixed proceeds rate L; and (iii) a short position in a finite-lived profit floor with a fixed proceeds rate H. Alternatively, the equivalent finite-lived price collar (34) can be understood as a portfolio composed by: (i) a long position in a finite-lived price cap capped at H; and (ii) a long position in a finite-lived profit floor with a fixed proceeds rate L.

4.3. Finite-lived price collars expressed in terms of portfolios of profit caps and profit floors

It is also straightforward to show that the payoff (28) is equivalent to

$$\Pi(P_t) = P_t - (P_t - H)\mathbb{1}_{\{P_t \geq H\}} + (L - P_t)\mathbb{1}_{\{P_t < L\}}. \tag{35}$$

Therefore, a finite-lived price collar can be expressed as

$$\begin{aligned} & C(P_0, L, H, T) \\ = & \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}[P_t]\, dt - \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(P_t - H)^+\right] dt + \int_0^T e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[(L - P_t)^+\right] dt \\ = & \frac{P_0}{q}\left(1 - e^{-qT}\right) - V(P_0, H, T) + F(P_0, L, T) && (36) \\ = & V_c(P_0, H, T) + F(P_0, L, T) && (37) \\ = & V_f(P_0, L, T) - V(P_0, H, T), && (38) \end{aligned}$$

with $V(P_0, H, T)$, $F(P_0, L, T)$, $V_c(P_0, H, T)$ and $V_f(P_0, L, T)$ being the finite maturity profit cap (4), the finite maturity profit floor (10), the finite maturity price cap (18) and the finite maturity price floor (22), respectively.

This implies that the finite-lived price collar (36) can be interpreted as a portfolio composed by: (i) the value of a plain active project with a duration equal to the time to maturity of the finite collar and having a present value equal to $\frac{P_0}{q}\left(1 - e^{-qT}\right)$; (ii) a short position in a finite-lived profit cap with a fixed cost rate H; and (iii) a long position in a finite-lived profit floor with a fixed proceeds rate L.

Remark 5 *The portfolio decomposition* (36) *is equivalent to the one that is offered in Adkins et al. (2019, equation 22 and Table 2), but only considering a finite concession with a duration equal to the maturity of the finite-lived collar, i.e., ignoring the value in operating the project after the end of the collar arrangement that is given by* $\frac{P_0}{q}e^{-qT}$. *We recall, however, that in our case it is not necessary to apply the replicating portfolio approach of Shackleton and Wojakowski (2007) to obtain the price collar* (36). *Moreover, the equivalent arbitrage-free relations* (30) *and* (33) *are expressed in terms of vertical bullish spreads of profit caps and* *profit floors, respectively, and offer an alternative economic intuition for the finite-lived price collar arrangement.*

Remark 6 *The portfolio decompositions* (37) *and* (38) *provide the rationale for two special cases of price collars that are often mentioned in the literature: (i) if $H \to \infty$, then* $\lim_{H\to\infty} V(P_0, H, T) = 0$ *because the price cap would never be exercised with an infinite strike price. This explains why a price collar arrangement with $H \to \infty$ is equivalent to a price floor after using the arbitrage-free relation* (38)*; and (ii) if $L \to 0$, then* $\lim_{L\to 0} F(P_0, L, T) = 0$ *because the price floor would never be exercised with an infinitesimal strike price. Again, this offers the intuition why a price collar arrangement with $L \to 0$ is equivalent to a price cap after using the arbitrage-free relation* (37)*.*

4.4. Finite-lived price collars expressed in terms of price floors or price caps

The finite-lived price collar (31) can be rearranged so that it is expressed in terms of price floors, that is

$$\begin{aligned} C(P_0, L, H, T) &= V_f(P_0, L, T) - V(P_0, H, T) - \frac{H}{r}\left(1 - e^{-rT}\right) + \frac{H}{r}\left(1 - e^{-rT}\right) \\ &= V_f(P_0, L, T) - V_f(P_0, H, T) + \frac{H}{r}\left(1 - e^{-rT}\right), \qquad (39) \end{aligned}$$

after using the arbitrage-free relation (21).

Similarly, the finite-lived price collar (34) can be rewritten in terms of price caps, that is

$$\begin{aligned} C(P_0, L, H, T) &= V_c(P_0, H, T) + F(P_0, L, T) - \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{L}{r}\left(1 - e^{-rT}\right) \\ &= \frac{L}{r}\left(1 - e^{-rT}\right) + V_c(P_0, H, T) - V_c(P_0, L, T), \qquad (40) \end{aligned}$$

after using the arbitrage-free relation (17).

4.5. Perpetual price collars

The corresponding perpetual price collars can be obtained by taking the limit $T \to \infty$ of any of the analytic representations presented above. Using, for example, the arbitrage-free relations (30), (33) and (36), then

$$\begin{aligned} C(P_0, L, H, \infty) &= \lim_{T\to\infty} C(P_0, L, H, T) \\ &= \frac{L}{r} + V(P_0, L, \infty) - V(P_0, H, \infty) \qquad (41) \\ &= \frac{H}{r} + F(P_0, L, \infty) - F(P_0, H, \infty) \qquad (42) \\ &= \frac{P_0}{q} - V(P_0, H, \infty) + F(P_0, L, \infty), \qquad (43) \end{aligned}$$

respectively.

Remark 7 *The arbitrage-free relations* (41), (42) *and* (43) *constitute viable alternatives for computing the value of an active project of a firm in the spirit of Adkins and Paxson (2019, equation 6) and Adkins et al. (2019, equation 9).*

5. Greeks

The so-called Greeks, also known as options sensitivity measures, are important tools for traders when implementing hedging and risk management strategies, especially in the case of naked short options positions. The computation of such Greek measures involves the partial derivative of the option price with respect to one of its input parameters. The most important Greeks are the delta, gamma and theta which correspond to the first and second derivatives with respect to the underlying asset price as well as time. These partial derivatives are especially relevant because they appear within the usual partial differential equation satisfied by the option price. Next proposition provides such Greek measures for a finite-lived cap.

Proposition 2 *Assume the lognormal process* (1) *with positive* r, q *and* σ, *as well as the finite-lived cap solution* (4).

(i) The delta of a finite-lived cap, $\Delta_V(P_0, X, T) := \partial V(P_0, X, T)/\partial P_0$, *is given by*

$$\begin{aligned} &\Delta_V(P_0, X, T) \\ = &-\frac{1}{q}\left[I(0,T,P_0,X,1,q) + P_0\, I_{P_0}(0,T,P_0,X,1,q)\right] + \frac{X}{r} I_{P_0}(0,T,P_0,X,-1,r), \end{aligned} \tag{44}$$

where

$$\begin{aligned} &I_{P_0}(0,T,P_0,X,\phi,v) \\ =: &\frac{\partial I(0,T,P_0,X,\phi,v)}{\partial P_0} \\ = &-\frac{1}{2\sigma P_0}\left(\frac{b_\phi}{c_v}+1\right)(c_v-b_\phi)\, e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-N(d(0))\right] \\ &+\frac{1}{2\sigma P_0}\left(\frac{b_\phi}{c_v}-1\right)(c_v+b_\phi)\, e^{-a\left(c_v+b_\phi\right)}\left[N\left(-\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-1+N(d(0))\right]. \end{aligned} \tag{45}$$

(ii) The gamma of a finite-lived cap, $\Gamma_V(P_0, X, T) := \partial^2 V(P_0, X, T)/\partial P_0^2$, *is given by*

$$\begin{aligned} &\Gamma_V(P_0, X, T) \\ = &-\frac{1}{q}\left[2\, I_{P_0}(0,T,P_0,X,1,q) + P_0 I_{P_0^2}(0,T,P_0,X,1,q)\right] + \frac{X}{r} I_{P_0^2}(0,T,P_0,X,-1,r), \end{aligned} \tag{46}$$

where

$$\begin{aligned} &I_{P_0^2}(0,T,P_0,X,\phi,v) \\ =: &\frac{\partial^2 I(0,T,P_0,X,\phi,v)}{\partial P_0^2} \\ = &\left(1-\frac{c_v-b_\phi}{\sigma}\right)\frac{1}{2\sigma P_0^2}\left(\frac{b_\phi}{c_v}+1\right)(c_v-b_\phi)\, e^{a\left(c_v-b_\phi\right)}\left[N\left(\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-N(d(0))\right] \\ &-\left(1+\frac{c_v+b_\phi}{\sigma}\right)\frac{1}{2\sigma P_0^2}\left(\frac{b_\phi}{c_v}-1\right)(c_v+b_\phi)\, e^{-a\left(c_v+b_\phi\right)} \\ &\times\left[N\left(-\frac{a}{\sqrt{T}}+c_v\sqrt{T}\right)-1+N(d(0))\right]. \end{aligned} \tag{47}$$

(iii) The theta of a finite-lived cap, $\theta_V(P_0, X, T) := \partial V(P_0, X, T)/\partial T$, *is given by*

$$\theta_V(P_0, X, T) = c_0(P_0, X, T), \tag{48}$$

where $c_0(P_0, X, t)$ *is the time-*0 *price of a European-style call option (or caplet) given in Black and Scholes (1973) and Merton (1973).*

Proof. These derivatives arise immediately after differentiating the finite-lived cap solution (4) with respect to the underlying asset price and time and by applying the chain rule to the cumulative normal distribution. Full details of the proof are available upon request. ■

In summary, the analytic representation of the Greeks presented in Proposition 2 constitute a viable alternative to the calculation of such partial derivatives obtained via the replicating portfolio approach of Shackleton and Wojakowski (2007). Even though Proposition 2 focuses only on the Greeks of finite-lived caps, the corresponding sensitivity measures of the other contractual arrangements discussed in this paper arise immediately using the aforementioned arbitrage-free relations.

6. Optimal investment decisions with price floors and price collars regimes

In this section, we analyze investment decisions with price floors and price collars regimes in the spirit of Barbosa et al. (2018) and Adkins et al. (2019) and we produce new and simpler formulae for calculating the optimal thresholds of investment. A similar rational can be applied in the real options applications considered in Barbosa et al. (2020), Barbosa et al. (2022) and Paxson et al. (2022).

6.1. Investment opportunity with a price floor regime

The goal now is to calculate the value of a (perpetual) option to invest in a renewable energy project to produce and sell energy with a price floor regime—also known as a feed-in-tariff (hereafter, FIT) contract—and the corresponding optimal investment rule.

6.1.1. Finite-lived case

Following Barbosa et al. (2018), we consider that after the expiry date of the finite maturity guarantee the producer of energy receives a cash flow with a present value equal to $\frac{P_0}{q}e^{-qT}$, that is understood as the value of selling the energy to the market after the maturity date of the FIT contract. Therefore, the total (or full) value of the project that includes the period of the FIT contract and the (perpetual) period thereafter is given by

$$\begin{aligned} V_f^F(P_0, L, T) &= V_f(P_0, L, T) + \frac{P_0}{q}e^{-qT} \\ &= V(P_0, L, T) + \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{P_0}{q}e^{-qT}, \end{aligned} \tag{49}$$

after using equation (21) that expresses the finite-lived price floor in terms of the profit cap (4). We recall that in the FIT contract under analysis the producer (or investor) is entitled to receive the price floor L for every unit of energy produced when the market price P_t is below the price floor; otherwise (i.e., when $P_t \geq L$), the producer receives the market price. Notice also that our definition (21) is equivalent to Barbosa et al. (2018, equation 19), though without the need of having different solutions for the regions $P_0 < L$ and $P_0 \geq L$.

It is well known that the value of the (perpetual) option to invest in a project with a total value as given in equation (49), $F_f(P_0, \overline{P}, L, T, K)$, with $\overline{P}$ being interpreted as the optimal price trigger level that induces the producer to invest in a project requiring an investment amount equal to K, is obtained as the solution of an ordinary differential equation subject to the usual boundary condition $\lim_{P_0 \to 0} F_f(P_0, \overline{P}, L, T, K) = 0$, thus yielding

$$F_f(P_0, \overline{P}, L, T, K) = AP_0^{\beta_1}, \tag{50}$$

where A is to be determined from appropriate value-matching and smooth-pasting conditions, while the constant parameter $\beta_1 (> 1)$ is given by

$$\begin{aligned} \beta_1 &= \frac{1}{2} - \frac{r-q}{\sigma^2} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ &= \frac{c_r - b_{-1}}{\sigma}. \end{aligned} \tag{51}$$

Applying the usual value-matching and smooth-pasting conditions to the option solution (50) at the optimal trigger $\overline{P}$, yields

$$\begin{cases} A\overline{P}^{\beta_1} & = V_f^F(\overline{P}, L, T) - K \\ \beta_1 A\overline{P}^{\beta_1 - 1} & = \frac{\partial}{\partial \overline{P}} V_f^F(\overline{P}, L, T) \end{cases}, \tag{52}$$

which is equivalent to

$$\begin{cases} \beta_1 A\overline{P}^{\beta_1} & = \beta_1 V_f^F(\overline{P}, L, T) - \beta_1 K \\ \beta_1 A\overline{P}^{\beta_1} & = \overline{P} \frac{\partial}{\partial \overline{P}} V_f^F(\overline{P}, L, T) \end{cases}, \tag{53}$$

after multiplying both sides of the value-matching and smooth-pasting conditions (52) by β_1 and $\overline{P}$, respectively.

Equating both branches of the system (53) yields

$$\beta_1 V_f^F(\overline{P}, L, T) - \beta_1 K = \overline{P} \frac{\partial}{\partial \overline{P}} V_f^F(\overline{P}, L, T), \tag{54}$$

which can be rewritten as

$$\begin{aligned} & \beta_1 \left(V(\overline{P}, L, T) + \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{\overline{P}}{q} e^{-qT} - K \right) \\ = \; & \overline{P} \frac{\partial}{\partial \overline{P}} \left(V(\overline{P}, L, T) + \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{\overline{P}}{q} e^{-qT} \right), \end{aligned} \tag{55}$$

by combining equations (49) and (54).

Finally, using equations (4) and (44) and applying straightforward calculus, the optimal trigger $\overline{P}$ can be determined by numerically solving the following nonlinear equation

$$\begin{aligned} & (\beta_1 - 1) \frac{\overline{P}}{q} \left[e^{-qT} - I(0, T, \overline{P}, L, 1, q) \right] + \beta_1 \left(\frac{L}{r} \left[1 - e^{-rT} + I(0, T, \overline{P}, L, -1, r) \right] - K \right) \\ & + \overline{P} \left[\frac{\overline{P}}{q} I_{\overline{P}}(0, T, \overline{P}, L, 1, q) - \frac{L}{r} I_{\overline{P}}(0, T, \overline{P}, L, -1, r) \right] = 0. \end{aligned} \tag{56}$$

Once the optimal trigger $\overline{P}$ is known we can immediately determine A using the value-matching condition (52) and replace it in equation (50), thus allowing the calculation of the value of the (perpetual) option to invest in the project as

$$F_f(P_0, \overline{P}, L, T, K) = \begin{cases} \left(V_f^F(\overline{P}, L, T) - K \right) \left(\frac{P_0}{\overline{P}} \right)^{\beta_1} & \Leftarrow \; P_0 < \overline{P} \\ V_f^F(P_0, L, T) - K & \Leftarrow \; P_0 \geq \overline{P} \end{cases}. \tag{57}$$

Notice that equation (56) is equivalent to Barbosa et al. (2018, equation 26), but with the great advantage that it is not necessary to know in advance in which region the threshold is, i.e., whether $\overline{P} < L$ or $\overline{P} \geq L$. In our case, we immediately know in which region the optimal trigger $\overline{P}$ is once equation (56) is solved.

By contrast, the use of Barbosa et al. (2018, equation 26) for determining the optimal threshold requires that first we need to find the investment cost level K_L^* that separates the regions $\overline{P} < L$ and $\overline{P} \geq L$. This is accomplished by solving Barbosa et al. (2018, equation 30) with respect to such investment cost level $K \equiv K_L^*$ and then compare it with the initial investment cost K. If $K_L^* \leq K$, the optimal threshold $\overline{P}$ is computed in the region $\overline{P} \geq L$ using the second branch of Barbosa et al. (2018, equation 26). Otherwise, i.e., if $K_L^* > K$, the optimal trigger $\overline{P}$ is calculated in the region $\overline{P} < L$ using the first branch of Barbosa et al. (2018, equation 26).

Remark 8 *There is still another alternative new method if one prefers to use the original formulae of finite-lived caps offered by Shackleton and Wojakowski (2007) that also requires only the numerical solution of a single nonlinear equation. In particular, substituting Shackleton and Wojakowski (2007, equations 21 and 28) in equation* (55) *and rearranging yields*

$$(\beta_1 - 1)\frac{\overline{P}}{q}\left[\mathbb{1}_{\{\overline{P}\geq L\}} + e^{-qT}N\left(-d_1\right)\right] + L\left(\frac{\overline{P}}{L}\right)^{\beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{q}\right)\left[\mathbb{1}_{\{\overline{P}\geq L\}} - N\left(d_{\beta_2}\right)\right]$$
$$+\beta_1\frac{L}{r}\left[\mathbb{1}_{\{\overline{P}<L\}} - e^{-rT}N\left(-d_0\right)\right] - \beta_1 K = 0, \tag{58}$$

with

$$\beta_{1,2} = \frac{1}{2} - \frac{r-q}{\sigma^2} \pm \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{59}$$

and

$$d_\beta = \frac{\ln(\overline{P}/L) + (r - q + (\beta - 0.5)\sigma^2)T}{\sigma\sqrt{T}}, \tag{60}$$

where $N(d_\beta)$ represents the usual cumulative density function of the univariate standard normal distribution for $\beta \in \{0, 1, \beta_2\}$. In summary, by using the arbitrage-free relation (49)*, we are able to simplify the calculation of the optimal threshold $\overline{P}$ via the new equation* (58)*, which is also equivalent to Barbosa et al. (2018, equation 26).*

Clearly, our approach for analyzing the optimal investment rule of a project with a price floor regime—whether using equations (56) or (58)—is much simpler than the one proposed by Barbosa et al. (2018).[2]

Nevertheless, and for the sake of completeness, we note that it is easy to determine analytically the investment cost level K_L^* that makes the current minimum price guarantee L the optimal trigger of investment. This is accomplished by replacing $\overline{P}$ by L in equation (56), that is

$$(\beta_1 - 1)\frac{L}{q}\left[e^{-qT} - I(0,T,L,L,1,q)\right] + \beta_1\left(\frac{L}{r}\left[1 - e^{-rT} + I(0,T,L,L,-1,r)\right] - K\right)$$
$$+L\left[\frac{L}{q}\,I_{\overline{P}=L}(0,T,L,L,1,q) - \frac{L}{r}\,I_{\overline{P}=L}(0,T,L,L,-1,r)\right] = 0. \tag{61}$$

For $\overline{P} = L$, it follows immediately that $a = 0$ and $N\left(d(0)\right) = 1/2$ after using equations (6) and (9), respectively. Therefore, equations (5) and (45) can be simplified to

$$I(0,T,L,L,\phi,v) = e^{-vT}N\left(b_\phi\sqrt{T}\right) - \frac{1}{2} - \frac{b_\phi}{c_v}\left[N\left(c_v\sqrt{T}\right) - \frac{1}{2}\right] \tag{62}$$

and

$$I_{\overline{P}=L}(0,T,L,L,\phi,v) = -\frac{2v}{\sigma L c_v}\left[N\left(c_v\sqrt{T}\right) - \frac{1}{2}\right], \tag{63}$$

respectively.

Finally, combining equations (61), (62) and (63) and rearranging the resulting terms we obtain a closed-form solution for determining the investment cost level $K \equiv K_L^*$ that makes the current minimum price guarantee L the optimal trigger of investment, that is

$$K_L^* = \frac{LY}{\beta_1}, \tag{64}$$

[2]Furthermore, we apply the value-matching and smooth-pasting conditions only one time to derive the investment threshold of a renewable energy project with a finite guarantee, whereas in Barbosa et al. (2018, Appendix C) the calculation of the optimal trigger in each region is derived separately, thus yielding two nonlinear equations. By the way, and for the sake of clarity, we notice that there is a very small typo in Barbosa et al. (2018, equation C.3). The correct formula is the one given in Barbosa et al. (2018, equation 26).

which is equivalent to Barbosa et al. (2018, equation 30), with

$$\begin{aligned} Y &= \frac{\beta_1 - 1}{q}\left(\frac{1}{2} + e^{-qT}\left[1 - N\left(b_1\sqrt{T}\right)\right] + \frac{b_1}{c_q}\left[N\left(c_q\sqrt{T}\right) - \frac{1}{2}\right]\right) \\ &+ \frac{\beta_1}{r}\left(\frac{1}{2} - e^{-rT}\left[1 - N\left(b_{-1}\sqrt{T}\right)\right] - \frac{b_{-1}}{c_r}\left[N\left(c_r\sqrt{T}\right) - \frac{1}{2}\right]\right) \\ &- \frac{2}{\sigma c_q}\left[N\left(c_q\sqrt{T}\right) - \frac{1}{2}\right] + \frac{2}{\sigma c_r}\left[N\left(c_r\sqrt{T}\right) - \frac{1}{2}\right]. \end{aligned} \tag{65}$$

There are other interesting points that are important from the perspective of the policy maker. For instance, taking now the initial investment cost K as given it is also possible to determine the L^* level that would be required to be offered by the government so that the firm starts the project immediately. This is accomplished by combining, again, equations (61), (62) and (63), but now solving the resulting expression with respect to $L \equiv L^*$, thus yielding the analytical solution

$$L^* = \frac{\beta_1 K}{Y}, \tag{66}$$

which, as expected, can be also derived from equation (64). If the policy maker sets the minimum price guarantee $L \geq L^*$, the investor starts immediately the project and receives a revenue from the guarantee instead of a revenue from the market price.

A further interesting point from the policymaker perspective is the price floor level L_0 that turns the net present value (henceforth, NPV) of the project equal to zero, because any value of L above this point generates a positive NPV independently of the market price P. Notice that if $L \geq L_0$ the investment will be deployed immediately because there is no waiting option. This point can be determined analytically by solving the equation NPV $:= \lim_{P_0 \to 0^+} V_f^F(P_0, L_0, T) - K = 0$ with respect to L_0 and with $V_f^F(P_0, L_0, T)$ being defined as the project value given in equation (49), which yields

$$L_0 = \frac{rK}{1 - e^{-rT}} \tag{67}$$

after applying straightforward calculus, thus being consistent, as expected, with Barbosa et al. (2018, equation 31).

There is another interesting point that, to the best of our knowledge, has not been mentioned in the context of optimal investment rules of projects with minimum price guarantees.

Let us now assume the minimum price guarantee L that is subject to the government policy. Even though the investment cost K is assumed to be fixed, it might be useful to know what would be the investment cost level that would make the current market price P_0 the trigger. In other words, what is the required investment cost, denoted hereafter by K^*, that induces the firm to invest immediately? We can answer this question if we replace $\overline{P}$ by P_0 in equation (56) and solve it with respect to the investment cost level $K \equiv K^*$, thus yielding the closed-form solution

$$\begin{aligned} K^* &= \frac{\beta_1 - 1}{\beta_1}\frac{P_0}{q}\left[e^{-qT} - I(0,T,P_0,L,1,q)\right] + \frac{L}{r}\left[1 - e^{-rT} + I(0,T,P_0,L,-1,r)\right] \\ &+ \frac{P_0}{\beta_1}\left[\frac{P_0}{q} I_{P_0}(0,T,P_0,L,1,q) - \frac{L}{r} I_{P_0}(0,T,P_0,L,-1,r)\right]. \end{aligned} \quad (68)$$

As expected, if $K \leq K^*$ it would be better for the firm to invest immediately because $P_0 = \overline{P}$ at the K^* level. Hence, when $K > K^*$, if the firm is able to reduce the investment cost from K to K^*—using, for instance, a cheaper technology that might be available in the market—, the current market price P_0 would be enough for the investor to exercise the option to invest immediately. Alternatively, for $K > K^*$, the amount $K - K^*$ can be understood as a subsidy value on investment that might be supported by the government if the goal is to deploy the investment immediately without the need of changing the policy with respect to the minimum price guarantee L. That is, the government supports the firm with a one-time subsidy amount that is paid upfront, while maintaining the fixed price floor L until the end of the finite-lived FIT scheme.

6.1.2. Perpetual case

Even though our main goal is to calculate the optimal investment triggers of the finite-lived case, the corresponding optimal thresholds for the perpetual case can be also determined from a nonlinear equation that is derived by taking the limit of equation (56) as $T \to \infty$, that is

$$\begin{aligned} &-(\beta_1 - 1)\frac{\overline{P}}{q} I(0,\infty,\overline{P},L,1,q) + \beta_1\left(\frac{L}{r}\left[1 + I(0,\infty,\overline{P},L,-1,r)\right] - K\right) \\ &+\overline{P}\left[\frac{\overline{P}}{q} I_{\overline{P}}(0,\infty,\overline{P},L,1,q) - \frac{L}{r} I_{\overline{P}}(0,\infty,\overline{P},L,-1,r)\right] = 0, \end{aligned} \quad (69)$$

where $I(0,\infty,\overline{P},L,\phi,v)$ is calculated via equation (12) and $I_{\overline{P}}(0,\infty,\overline{P},L,\phi,v)$ is obtained by taking the limit of equation (45) as $T\to\infty$, yielding

$$\begin{aligned} I_{\overline{P}}(0,\infty,\overline{P},L,\phi,v) &= -\frac{1}{2\sigma P_0}\left(\frac{b_\phi}{c_v}+1\right)(c_v-b_\phi)\,e^{a\left(c_v-b_\phi\right)}\left[1-N\left(d(0)\right)\right] \\ &\quad +\frac{1}{2\sigma P_0}\left(\frac{b_\phi}{c_v}-1\right)(c_v+b_\phi)\,e^{-a\left(c_v+b_\phi\right)}N\left(d(0)\right). \end{aligned} \tag{70}$$

Notice that equation (69) is an alternative to Barbosa et al. (2018, equation 16) for computing the optimal triggers for a project with a perpetual minimum price guarantee L. As usual, equation (69) is valid only when $\overline{P}\geq L$, because the value-matching and smooth-pasting conditions are not accommodated in the region $\overline{P}<L$ for the perpetual case. Adopting a similar procedure to one employed in the finite-lived case, we can determine the value of the (perpetual) option to invest in the project once the optimal trigger $\overline{P}$ of the perpetual case is known, that is

$$F_f(P_0,\overline{P},L,\infty,K)=\begin{cases}\left(V_f^F(\overline{P},L,\infty)-K\right)\left(\frac{P_0}{\overline{P}}\right)^{\beta_1} & \Leftarrow\quad P_0<\overline{P} \\ V_f^F(P_0,L,\infty)-K & \Leftarrow\quad P_0\geq\overline{P}\end{cases}, \tag{71}$$

where $V_f^F(P,L,\infty)$ is computed as the limit of equation (49) as $T\to\infty$, which yields the perpetual price floor solution (23).

Moreover, it is also possible to calculate the interesting point where the trigger $\overline{P}$ is equal to the price floor $L\equiv L_\infty^*$ with the nonlinear equation (69) being rewritten as

$$\begin{aligned} &-\left(\beta_1-1\right)\frac{L}{q}I(0,\infty,L,L,1,q)+\beta_1\left(\frac{L}{r}\left[1+I(0,\infty,L,L,-1,r)\right]-K\right) \\ &+L\left[\frac{L}{q}\,I_{\overline{P}=L}(0,\infty,L,L,1,q)-\frac{L}{r}\,I_{\overline{P}=L}(0,\infty,L,L,-1,r)\right]=0. \end{aligned} \tag{72}$$

Taking the limits of equations (62) and (63) as $T\to\infty$, it follows that

$$I(0,\infty,L,L,\phi,v)=-\frac{1}{2}\left(1+\frac{b_\phi}{c_v}\right) \tag{73}$$

and

$$I_{\overline{P}=L}(0,\infty,L,L,\phi,v)=-\frac{v}{\sigma L c_v}. \tag{74}$$

Finally, combining equations (72), (73) and (74) it is possible to show that

$$L_\infty^*=rK, \tag{75}$$

after applying straightforward calculus, which is equivalent to Barbosa et al. (2018, equation 18). This allows us to determine also the required investment cost that makes the fixed L the optimal trigger of investment, i.e., $K \equiv K_L^* = L/r$. In summary and assuming the given initial investment cost K, if $K_L^* \leq K$, the optimal threshold $\overline{P}$ can be calculated since $\overline{P} \geq L$; otherwise, i.e., if $K_L^* > K$, the value-matching and smooth-pasting conditions are not met under the perpetual case and, hence, equation (69) is not valid.

Two additional results can be now provided combining some of the finite maturity and perpetual solutions. For instance, comparing equations (66) and (75) allows us to offer the asymptotic result $\lim_{T\to\infty} \beta_1/Y = r$, which can be easily proved analytically. Furthermore, combining equations (67) and (75) yields

$$L_0 \left(1 - e^{-rT}\right) = L_\infty^*, \tag{76}$$

which is consistent with the observation of Barbosa et al. (2018) that if the government sets a policy with a price floor $L \geq L_0$ in a finite maturity contract or a price floor $L \geq L_\infty^*$ in a perpetual contract, the investment will be made immediately generating a risk-free profit. Hence, we would have the counter-intuitive result of a project generating a positive NPV independently of the market price P in both FIT schemes.

6.2. Investment opportunity with a price collar regime

The purpose now is to show how to calculate the value of a (perpetual) option to invest in a project with a collar-style arrangement and the corresponding optimal investment rule.

6.2.1. Finite-lived case

Similarly to Adkins et al. (2019), we assume that after the expiry date of the collar-style arrangement the investor receives a cash flow with a present value equal to $\frac{P_0}{q}e^{-qT}$ that captures the value of operating the project perpetually without restrictions on the market price after the maturity date of the collar agreement. Hence, the total (or full) value of the project that includes the period of the collar arrangement and the (perpetual) period thereafter is given by

$$\begin{aligned} C^F(P_0, L, H, T) &= C(P_0, L, H, T) + \frac{P_0}{q} e^{-qT} \\ &= V(P_0, L, T) - V(P_0, H, T) + \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{P_0}{q} e^{-qT}, \end{aligned} \tag{77}$$

after using equation (30) that expresses the finite-lived collar in terms of the profit cap (4). Notice that our definition (77) is equivalent to Adkins et al. (2019, equation 22), though without the need of having different solutions for the regions $P_0 < L$, $L \le P_0 < H$ and $P_0 \ge H$.

The value of the (perpetual) option to invest in a project with a total value as given in equation (77), $F_C(P_0, \overline{P}, L, H, T, K)$, with $\overline{P}$ being interpreted again the optimal price trigger level inducing the producer to invest in a project requiring an investment cost K, is also obtained as the solution of an ordinary differential equation subject to the boundary condition $\lim_{P_0 \to 0} F_C(P_0, \overline{P}, L, H, T, K) = 0$, which yields

$$F_C(P_0, \overline{P}, L, H, T, K) = A P_0^{\beta_1}, \tag{78}$$

where A needs to be determined from appropriate value-matching and smooth-pasting conditions, whereas the constant parameter $\beta_1 (> 1)$ is still given by equation (51).

Applying the value-matching and smooth-pasting conditions to the option solution (78) at the optimal trigger $\overline{P}$, yields

$$\left\{ \begin{aligned} A\overline{P}^{\beta_1} &= C^F(\overline{P}, L, H, T) - K \\ \beta_1 A \overline{P}^{\beta_1 - 1} &= \tfrac{\partial}{\partial \overline{P}} C^F(\overline{P}, L, H, T) \end{aligned} \right. , \tag{79}$$

which is equivalent to

$$\left\{ \begin{aligned} \beta_1 A\overline{P}^{\beta_1} &= \beta_1 C^F(\overline{P}, L, H, T) - \beta_1 K \\ \beta_1 A \overline{P}^{\beta_1} &= \overline{P} \tfrac{\partial}{\partial \overline{P}} C^F(\overline{P}, L, H, T) \end{aligned} \right. , \tag{80}$$

after multiplying both sides of the value-matching and smooth-pasting conditions (79) by β_1 and $\overline{P}$, respectively.

Equating both branches of the system (80) yields

$$\beta_1 C^F(\overline{P}, L, H, T) - \beta_1 K = \overline{P} \frac{\partial}{\partial \overline{P}} C^F(\overline{P}, L, H, T), \tag{81}$$

which can be rewritten as

$$\begin{aligned} & \beta_1 \left(V(\overline{P}, L, T) - V(\overline{P}, H, T) + \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{\overline{P}}{q} e^{-qT} - K \right) \\ = \; & \overline{P} \frac{\partial}{\partial \overline{P}} \left(V(\overline{P}, L, T) - V(\overline{P}, H, T) + \frac{L}{r}\left(1 - e^{-rT}\right) + \frac{\overline{P}}{q} e^{-qT} \right), \end{aligned} \tag{82}$$

by combining equations (77) and (81).

Finally, using equations (4) and (44) and applying straightforward calculus, the optimal trigger $\overline{P}$ can be determined by numerically solving the following nonlinear equation

$$\begin{aligned} & (\beta_1 - 1) \frac{\overline{P}}{q} \left[e^{-qT} + \tilde{I}(T, 1, q) \right] + \beta_1 \left[\frac{L}{r}\left(1 - e^{-rT}\right) - \frac{1}{r} \tilde{I}(T, -1, r) - K \right] \\ & - \frac{\overline{P}^2}{q} \tilde{I}_{\overline{P}}(T, 1, q) + \frac{\overline{P}}{r} \tilde{I}_{\overline{P}}(T, -1, r) = 0, \end{aligned} \tag{83}$$

where

$$\tilde{I}(T, 1, q) := I(0, T, \overline{P}, H, 1, q) - I(0, T, \overline{P}, L, 1, q), \tag{84}$$

$$\tilde{I}(T, -1, r) := H \times I(0, T, \overline{P}, H, -1, r) - L \times I(0, T, \overline{P}, L, -1, r), \tag{85}$$

$$\tilde{I}_{\overline{P}}(T, 1, q) := I_{\overline{P}}(0, T, \overline{P}, H, 1, q) - I_{\overline{P}}(0, T, \overline{P}, L, 1, q) \tag{86}$$

and

$$\tilde{I}_{\overline{P}}(T, -1, r) := H \times I_{\overline{P}}(0, T, \overline{P}, H, -1, r) - L \times I_{\overline{P}}(0, T, \overline{P}, L, -1, r). \tag{87}$$

We note that we obtain the required investment trigger for the case of finite-lived collars by simply solving equation (83). Hence, it is not necessary to know in advance in which region the threshold is, i.e., whether $\overline{P} > H$, $L \leq \overline{P} < H$ or $\overline{P} < L$. Similarly to the FIT case, we immediately know in which region the optimal trigger $\overline{P}$ is once equation (83) is solved.

By contrast, the procedure proposed by Adkins et al. (2019, Appendix B) for determining the optimal threshold requires first the determination of the region in which the threshold $\overline{P}$

is. Clearly, our approach for analyzing the optimal investment rule of a project with a price collar regime is much simpler than the one offered by Adkins et al. (2019).[3]

Remark 9 *There is still another alternative new method if one prefers to use the original formulae of finite-lived caps offered by Shackleton and Wojakowski (2007) that also requires only the numerical solution of a single nonlinear equation. In particular, substituting Shackleton and Wojakowski (2007, equations 21 and 28) in equation* (82) *and rearranging yields*

$$\begin{aligned}
&(\beta_1 - 1)\frac{\overline{P}}{q}\left[\mathbb{1}_{\{L \le \overline{P} < H\}} + e^{-qT}\left(N\left(-d_1^L\right) + N\left(d_1^H\right)\right)\right] \\
&+\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{q}\right)\left[L\left(\frac{\overline{P}}{L}\right)^{\beta_2}\left(\mathbb{1}_{\{\overline{P} \ge L\}} - N\left(d_{\beta_2}^L\right)\right) - H\left(\frac{\overline{P}}{H}\right)^{\beta_2}\left(\mathbb{1}_{\{\overline{P} \ge H\}} - N\left(d_{\beta_2}^H\right)\right)\right] \\
&+\beta_1\left[\frac{L}{r}\left(\mathbb{1}_{\{\overline{P} < L\}} - e^{-rT}N\left(-d_0^L\right)\right) + \frac{H}{r}\left(\mathbb{1}_{\{\overline{P} \ge H\}} - e^{-rT}N\left(d_0^H\right)\right)\right] - \beta_1 K = 0, \qquad (88)
\end{aligned}$$

with $\beta_{1,2}$ *being still given by equation* (59)*, while*

$$d_\beta^X = \frac{\ln(\overline{P}/X) + (r - q + (\beta - 0.5)\sigma^2)T}{\sigma\sqrt{T}}, \qquad (89)$$

where $N(d_\beta^X)$ *represents the usual cumulative density function of the univariate standard normal distribution for* $\beta \in \{0, 1, \beta_2\}$ *and* $X \in \{L, H\}$*. In summary, by using the arbitrage-free relation* (77)*, we are able to simplify the calculation of the optimal threshold* $\overline{P}$ *via the new equation* (88)*, which is also equivalent and much simpler than the methodology procedure proposed by Adkins et al. (2019, Appendix B) since only a single nonlinear equation needs to be numerically solved.*

[3] Moreover, we apply the value-matching and smooth-pasting conditions only one time when deriving the investment threshold of a project with a finite-lived collar-style arrangement, whereas in Adkins et al. (2019) such boundary conditions are applied individually, i.e., for each one of the three possible regions, thus yielding three nonlinear equations. By the way, and for the sake of clarity, we notice that there are some small typos in Adkins et al. (2019, equation 53). The correct formula is the one given in Adkins et al. (2019, equation 23).

Once the optimal trigger $\overline{P}$ is known we are able to determine A using the value-matching condition (79). Finally, replacing the obtained A in equation (78) allows us to calculate the value of the (perpetual) option to invest in the project as

$$F_C(P_0, \overline{P}, L, H, T, K) = \begin{cases} \left(C^F(\overline{P}, L, H, T) - K\right)\left(\frac{P_0}{\overline{P}}\right)^{\beta_1} & \Leftarrow \quad P_0 < \overline{P} \\ C^F(P_0, L, H, T) - K & \Leftarrow \quad P_0 \geq \overline{P} \end{cases}. \tag{90}$$

6.2.2. Perpetual case

Although the main purpose here is not to tackle the infinite-horizon problem, the optimal thresholds for the perpetual case can be also determined from a nonlinear equation that is derived by taking the limit of equation (83) as $T \to \infty$, that is

$$\begin{aligned} &(\beta_1 - 1)\,\frac{\overline{P}}{q}\tilde{I}(\infty, 1, q) + \beta_1\left[\frac{L}{r} - \frac{1}{r}\,\tilde{I}(\infty, -1, r) - K\right] \\ &-\frac{\overline{P}^2}{q}\,\tilde{I}_{\overline{P}}(\infty, 1, q) + \frac{\overline{P}}{r}\,\tilde{I}_{\overline{P}}(\infty, -1, r) = 0, \end{aligned} \tag{91}$$

where

$$\tilde{I}(\infty, 1, q) := I(0, \infty, \overline{P}, H, 1, q) - I(0, \infty, \overline{P}, L, 1, q), \tag{92}$$

$$\tilde{I}(\infty, -1, r) := H \times I(0, \infty, \overline{P}, H, -1, r) - L \times I(0, \infty, \overline{P}, L, -1, r), \tag{93}$$

$$\tilde{I}_{\overline{P}}(\infty, 1, q) := I_{\overline{P}}(0, \infty, \overline{P}, H, 1, q) - I_{\overline{P}}(0, \infty, \overline{P}, L, 1, q) \tag{94}$$

and

$$\tilde{I}_{\overline{P}}(\infty, -1, r) := H \times I_{\overline{P}}(0, \infty, \overline{P}, H, -1, r) - L \times I_{\overline{P}}(0, \infty, \overline{P}, L, -1, r). \tag{95}$$

Notice that our approach allows us to determine the optimal trigger $\overline{P}$ by simply solving equation (91), whereas the alternative that is available in the literature requires a more complex find-root scheme involving three equations as shown in Adkins et al. (2019, equations 18, 19 and 20), because they need to find first in which region the optimal threshold is placed.

Finally, using a similar procedure to one employed in the finite-lived case, we can determine the value of the (perpetual) option to invest in the project once the optimal trigger $\overline{P}$ of the perpetual case is known, that is

$$F_C(P_0, \overline{P}, L, H, \infty, K) = \begin{cases} \left(C^F(\overline{P}, L, H, \infty) - K\right)\left(\frac{P_0}{\overline{P}}\right)^{\beta_1} & \Leftarrow \quad P_0 < \overline{P} \\ C^F(P_0, L, H, \infty) - K & \Leftarrow \quad P_0 \geq \overline{P} \end{cases}, \tag{96}$$

where $C^F(P, L, H, \infty)$ is computed as the limit of equation (77) as $T \to \infty$, which yields the perpetual price collar solution (41).

To sum up, the novel solutions proposed in this section allow us to evaluate optimal decisions of investment projects with a minimum price guarantee (i.e., a price floor regime) and with a price collar regime that have been recently analyzed in Barbosa et al. (2018) and Adkins et al. (2019), respectively. In particular, the results reported by these authors (e.g., tables and figures) can now be immediately replicated using a viable alternative approach that is very simple and easier to implement in any software package.

7. Conclusions

The valuation of finite maturity caps, floors and collars on continuous flows has been typically tackled using perpetual methods inspired in the real options literature. Under this approach, the perpetual solution of the problem needs to be solved first and only afterwards the more interesting finite maturity case is deduced by subtracting the risk-neutral expectation of the forward start perpetual solution from the corresponding (current) perpetual solution.

By contrast, we propose a novel approach that does not require the use of the replicating portfolio scheme of Shackleton and Wojakowski (2007), since we are able to directly analyze the finite maturity case by solving the finite time integral through a relatively simple integration by parts method. Whenever necessary, the corresponding perpetual claims arise immediately by taking $T \to \infty$ in the finite maturity solutions.

Several arbitrage-free relations between different, but related, contractual arrangements and the so-called Greeks are also derived in closed-form. Moreover, we provide two new methods for calculating the optimal triggers of investment projects containing price floors and price collars clauses that are shown to be very simple and easy to implement, even when used in general-purpose computer programs such as spreadsheets.

Appendix A: Proof of Proposition 1

This appendix is organized in four parts:

(i) Combining equation (2) with the closed-form solution of a plain-vanilla call (or caplet) offered in Black and Scholes (1973) and Merton (1973), the finite-lived cap can be rewritten as

$$\begin{aligned} V(P_0, X, T) &= \int_0^T c_0\,(P_0, X, t)\,dt \\ &= \int_0^T \left[P_0 e^{-qt} N(d_1(t)) - X e^{-rt} N(d_2(t))\right] dt \\ &= -\frac{P_0}{q}\int_0^T (-q)e^{-qt}N(d_1(t))dt + \frac{X}{r}\int_0^T (-r)e^{-rt}N(d_2(t))dt \\ &= -\frac{P_0}{q}I_1 + \frac{X}{r}I_2 \end{aligned} \tag{.1}$$

with

$$d_1(t) := \frac{\ln(P_0/X) + (r - q + \sigma^2/2)t}{\sigma\sqrt{t}}, \tag{.2}$$

$$d_2(t) := \frac{\ln(P_0/X) + (r - q - \sigma^2/2)t}{\sigma\sqrt{t}}, \tag{.3}$$

$$I_1 := \int_0^T (-q)e^{-qt}N(d_1(t))dt, \tag{.4}$$

$$I_2 := \int_0^T (-r)e^{-rt}N(d_2(t))dt \tag{.5}$$

and where $N(.)$ represents the cumulative density function of the univariate standard normal distribution.

The integrals I_1 and I_2 are similar and can be both solved using integration by parts. Defining $a_1 := \ln(P_0/X)/\sigma$ and $b_1 := (r - q + \sigma^2/2)/\sigma$, equation (.2) can be rewritten as

$$d_1(t) = \frac{a_1}{\sqrt{t}} + b_1\sqrt{t}. \tag{.6}$$

Applying integration by parts to the first integral and recalling that the probability density function of the standard normal distribution is given by $n(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$, then

$$\begin{aligned}
I_1 &= e^{-qt} N\left(a_1 t^{-1/2} + b_1 t^{1/2}\right)\Big|_{t=0}^{t=T} \\
&\quad - \int_0^T e^{-qt}\, n\left(a_1 t^{-1/2} + b_1 t^{1/2}\right)\left(\frac{-a_1 t^{-3/2} + b_1 t^{-1/2}}{2}\right) dt \\
&= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - e^0 N\left(d_1(0)\right) \\
&\quad - \int_0^T e^{-qt} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(a_1 t^{-1/2} + b_1 t^{1/2}\right)^2}{2}\right)\left(\frac{-a_1 t^{-3/2} + b_1 t^{-1/2}}{2}\right) dt \\
&= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
&\quad - \int_0^T e^{-qt} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a_1^2 t^{-1} + 2a_1 b_1 + b_1^2 t}{2}\right)\left(\frac{-a_1 t^{-3/2} + b_1 t^{-1/2}}{2}\right) dt \\
&= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
&\quad - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\left(b_1^2 + 2q\right)t + a_1^2 t^{-1}\right)}\left(\frac{b_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2}\right) dt \\
&= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
&\quad - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)}\left(\frac{b_1}{c_1}\frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2}\right) dt,
\end{aligned} \tag{.7}$$

with $c_1 := \sqrt{b_1^2 + 2q}$ and

$$\begin{aligned}
N\left(d_1(0)\right) &:= N\left(\lim_{t\to 0}\left(a_1 t^{-1/2} + b_1 t^{1/2}\right)\right) \\
&= \begin{cases} N(+\infty) & \Leftarrow \; a_1 > 0 \\ N(0) & \Leftarrow \; a_1 = 0 \\ N(-\infty) & \Leftarrow \; a_1 < 0 \end{cases}. \\
&= \begin{cases} 1 & \Leftarrow \; a_1 > 0 \\ \frac{1}{2} & \Leftarrow \; a_1 = 0 \\ 0 & \Leftarrow \; a_1 < 0 \end{cases}. \\
&= 1 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 0 \times \mathbb{1}_{\{P_0 < X\}}.
\end{aligned} \tag{.8}$$

Noting that

$$
\begin{aligned}
& \frac{b_1}{c_1}\frac{c_1}{2}t^{-1/2} - \frac{a_1}{2}t^{-3/2} \\
= & \frac{b_1}{c_1}\frac{c_1}{2}t^{-1/2} + \frac{c_1}{2}t^{-1/2} - \frac{c_1}{2}t^{-1/2} - \frac{a_1}{2}t^{-3/2} + \frac{b_1}{c_1}\frac{a_1}{2}t^{-3/2} - \frac{b_1}{c_1}\frac{a_1}{2}t^{-3/2} \\
= & \frac{1}{2}\left(\frac{b_1}{c_1} + 1\right)\left(\frac{c_1}{2}t^{-1/2} - \frac{a_1}{2}t^{-3/2}\right) + \frac{1}{2}\left(\frac{b_1}{c_1} - 1\right)\left(\frac{c_1}{2}t^{-1/2} + \frac{a_1}{2}t^{-3/2}\right),
\end{aligned}
$$

then the integral (.7) can be rewritten as

$$
\begin{aligned}
I_1 = \; & e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
& - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)} \frac{1}{2}\left(\frac{b_1}{c_1} + 1\right)\left(\frac{c_1}{2}t^{-1/2} - \frac{a_1}{2}t^{-3/2}\right) dt \\
& - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)} \frac{1}{2}\left(\frac{b_1}{c_1} - 1\right)\left(\frac{c_1}{2}t^{-1/2} + \frac{a_1}{2}t^{-3/2}\right) dt. \qquad (.9)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
e^{-a_1 b_1} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)} &= e^{-a_1 b_1} e^{-c_1 a_1} e^{c_1 a_1} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)} \\
&= e^{-a_1 b_1} e^{c_1 a_1} e^{-\frac{1}{2}\left(c_1^2 t + 2 c_1 a_1 + a_1^2 t^{-1}\right)} \\
&= e^{a_1 (c_1 - b_1)} e^{-\frac{1}{2}\left(c_1 t^{1/2} + a_1 t^{-1/2}\right)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-a_1 b_1} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)} &= e^{-a_1 b_1} e^{-c_1 a_1} e^{c_1 a_1} e^{-\frac{1}{2}\left(c_1^2 t + a_1^2 t^{-1}\right)} \\
&= e^{-a_1 b_1} e^{-c_1 a_1} e^{-\frac{1}{2}\left(c_1^2 t - 2 c_1 a_1 + a_1^2 t^{-1}\right)} \\
&= e^{-a_1 (c_1 + b_1)} e^{-\frac{1}{2}\left(c_1 t^{1/2} - a_1 t^{-1/2}\right)^2},
\end{aligned}
$$

which implies that the integral (.9) can be rewritten as

$$
\begin{aligned}
I_1 &= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}+1\right) e^{a_1(c_1-b_1)} \int_0^T \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(c_1 t^{1/2}+a_1 t^{-1/2}\right)^2} \left(\frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2}\right) dt \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}-1\right) e^{-a_1(c_1+b_1)} \int_0^T \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(c_1 t^{1/2}-a_1 t^{-1/2}\right)^2} \left(\frac{c_1}{2} t^{-1/2} + \frac{a_1}{2} t^{-3/2}\right) dt \\
&= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}+1\right) e^{a_1(c_1-b_1)} \int_0^T \left[\frac{d}{dt} N\left(c_1 t^{1/2} + a_1 t^{-1/2}\right)\right] dt \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}-1\right) e^{-a_1(c_1+b_1)} \int_0^T \left[\frac{d}{dt} N\left(c_1 t^{1/2} - a_1 t^{-1/2}\right)\right] dt \\
&= e^{-qT} N\left(a_1 T^{-1/2} + b_1 T^{1/2}\right) - N\left(d_1(0)\right) \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}+1\right) e^{a_1(c_1-b_1)} N\left(c_1 t^{1/2} + a_1 t^{-1/2}\right)\Big|_{t=0}^{t=T} \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}-1\right) e^{-a_1(c_1+b_1)} N\left(c_1 t^{1/2} - a_1 t^{-1/2}\right)\Big|_{t=0}^{t=T} \\
&= e^{-qT} N\left(\frac{a_1}{\sqrt{T}} + b_1\sqrt{T}\right) - N\left(d_1(0)\right) \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}+1\right) e^{a_1(c_1-b_1)} \left[N\left(\frac{a_1}{\sqrt{T}} + c_1\sqrt{T}\right) - N\left(d_1(0)\right)\right] \\
&\quad -\frac{1}{2}\left(\frac{b_1}{c_1}-1\right) e^{-a_1(c_1+b_1)} \left[N\left(-\frac{a_1}{\sqrt{T}} + c_1\sqrt{T}\right) - 1 + N\left(d_1(0)\right)\right], \qquad (.10)
\end{aligned}
$$

because, following the insights used for calculating the distribution (.8), it is possible to show that

$$
\begin{aligned}
N\left(\lim_{t\to 0}\left(c_1 t^{1/2} + a_1 t^{-1/2}\right)\right) &= 1 \times 1\!\!1_{\{P_0>X\}} + \frac{1}{2} \times 1\!\!1_{\{P_0=X\}} + 0 \times 1\!\!1_{\{P_0<X\}} \\
&= N\left(d_1(0)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
N\left(\lim_{t\to 0}\left(c_1 t^{1/2} - a_1 t^{-1/2}\right)\right) &= 0 \times 1\!\!1_{\{P_0>X\}} + \frac{1}{2} \times 1\!\!1_{\{P_0=X\}} + 1 \times 1\!\!1_{\{P_0<X\}} \\
&= 1 - N\left(d_1(0)\right).
\end{aligned}
$$

Applying the same rational to the second integral with $a_2 := \ln(P_0/X)/\sigma$, $b_2 := (r - q - \sigma^2/2)/\sigma$ and $c_2 := \sqrt{b_2^2 + 2r}$, it can be shown that

$$\begin{aligned} I_2 &= e^{-rT} N\left(\frac{a_2}{\sqrt{T}} + b_2\sqrt{T}\right) - N\left(d_2(0)\right) \\ &\quad -\frac{1}{2}\left(\frac{b_2}{c_2} + 1\right) e^{a_2(c_2 - b_2)} \left[N\left(\frac{a_2}{\sqrt{T}} + c_2\sqrt{T}\right) - N\left(d_2(0)\right)\right] \\ &\quad -\frac{1}{2}\left(\frac{b_2}{c_2} - 1\right) e^{-a_2(c_2 + b_2)} \left[N\left(-\frac{a_2}{\sqrt{T}} + c_2\sqrt{T}\right) - 1 + N\left(d_2(0)\right)\right], \end{aligned} \tag{.11}$$

with $N\left(d_2(0)\right) = 1 \times 1\!\!1_{\{P_0 > X\}} + \frac{1}{2} \times 1\!\!1_{\{P_0 = X\}} + 0 \times 1\!\!1_{\{P_0 < X\}} = N\left(d_1(0)\right)$.

Finally, equation (4) arises immediately from equation (.1) since the integrals (.10) and (.11) can be expressed in a compact form as shown in equation (5).

(ii) Combining equation (3) with the closed-form solution of a plain-vanilla put (or floorlet) offered in Black and Scholes (1973) and Merton (1973), the finite-lived floors can be rewritten as

$$\begin{aligned} & F(P_0, X, T) \\ = & \int_0^T p_0\left(P_0, X, t\right) dt \\ = & \int_0^T \left[X e^{-rt} N(-d_2(t)) - P_0 e^{-qt} N(-d_1(t))\right] dt \\ = & \int_0^T X e^{-rt} dt - \int_0^T X e^{-rt} N(d_2(t)) dt - \int_0^T P_0 e^{-qt} dt + \int_0^T P_0 e^{-qt} N(d_1(t)) dt \\ = & -\frac{X}{r} \int_0^T (-r) e^{-rt} dt + \frac{X}{r} \int_0^T (-r) e^{-rt} N(d_2(t)) dt \\ & + \frac{P_0}{q} \int_0^T (-q) e^{-qt} dt - \frac{P_0}{q} \int_0^T (-q) e^{-qt} N(d_1(t)) dt \\ = & -\frac{X}{r} \left. e^{-rt} \right|_{t=0}^{t=T} + \frac{X}{r} I_2 + \frac{P_0}{q} \left. e^{-qt} \right|_{t=0}^{t=T} - \frac{P_0}{q} I_1 \\ = & \frac{X}{r}\left(1 - e^{-rT}\right) - \frac{P_0}{q}\left(1 - e^{-qT}\right) + \frac{X}{r} I_2 - \frac{P_0}{q} I_1, \end{aligned} \tag{.12}$$

which finally yields equation (10) after replacing the integrals (.10) and (.11) and equation (4) in equation (.12).

(iii) The perpetual cap is defined as

$$\begin{aligned} V(P_0, X, \infty) &= \int_0^\infty c_0\,(P_0, X, t)\,dt \\ &= \lim_{T\to\infty} \int_0^T c_0\,(P_0, X, t)\,dt \\ &= \lim_{T\to\infty} V(P_0, X, T), \end{aligned} \tag{.13}$$

with $V(P_0, X, T)$ being the finite-lived cap solution given in equation (4). The analytical representation of the perpetual cap can then be obtained after evaluating the integral

$$I(0, \infty, P_0, X, \phi, v) := \lim_{T\to\infty} I(0, T, P_0, X, \phi, v), \tag{.14}$$

with $I(0, T, P_0, X, \phi, v)$ being the integral defined in equation (5).

To accomplish this purpose three limits must be evaluated. Since r and q are both positive then $v > 0$ and, therefore,

$$\lim_{T\to\infty} e^{-vT} N\left(\frac{a}{\sqrt{T}} + b\sqrt{T}\right) = 0, \tag{.15}$$

because e^{-vT} is an infinitesimal, $N\,(.)$ is a limited function in $[0, 1]$ and, hence, its product is an infinitesimal. Regarding the other two limits, and because $c > 0$,

$$\begin{aligned} \lim_{T\to\infty} N\left(\frac{a}{\sqrt{T}} + c\sqrt{T}\right) &= N\,(+\infty) \\ &= 1 \end{aligned} \tag{.16}$$

and

$$\begin{aligned} \lim_{T\to\infty} N\left(-\frac{a}{\sqrt{T}} + c\sqrt{T}\right) &= N\,(+\infty) \\ &= 1. \end{aligned} \tag{.17}$$

The integral $I(0, \infty, P_0, X, \phi, v)$ shown in equation (12) is then obtained after combining the limits (.14), (.15), (.16) and (.17), which finally allows the computation of the perpetual cap shown in equation (11).

(iv) The perpetual floor is defined as

$$\begin{aligned} F(P_0, X, \infty) &= \int_0^\infty p_0\,(P_0, X, t)\, dt \\ &= \lim_{T\to\infty} \int_0^T p_0\,(P_0, X, t)\, dt \\ &= \lim_{T\to\infty} F(P_0, X, T), \end{aligned} \tag{.18}$$

with $F(P_0, X, T)$ being the finite-lived floor solution given in equation (10). Finally, equation (13) arises immediately after computing the limit (.18).

References

Adkins, Roger and Dean Paxson, 2019, Real Collars as Alternative Incentives for Subsidizing Energy Facilities, *Manchester School* 87, 428–454.

Adkins, Roger, Dean Paxson, Paulo J. Pereira, and Artur Rodrigues, 2019, Investment Decisions with Finite-Lived Collars, *Journal of Economic Dynamics and Control* 103, 185–204.

Barbosa, Luciana, Artur Rodrigues, and Alberto Sardinha, 2022, Optimal Price Subsidies under Uncertainty, *European Journal of Operational Research* 303, 471–479.

Barbosa, Luciana, Cláudia Nunes, Artur Rodrigues, and Alberto Sardinha, 2020, Feed-In Tariff Contract Schemes and Regulatory Uncertainty, *European Journal of Operational Research* 287, 331–347.

Barbosa, Luciana, Paulo Ferrão, Artur Rodrigues, and Alberto Sardinha, 2018, Feed-In-Tariffs with Minimum Price Guarantees and Regulatory Uncertainty, *Energy Economics* 72, 517–541.

Black, Fischer, 1976, The Pricing of Commodity Contracts, *Journal of Financial Economics* 3, 167–179.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Boyle, Phelim P. and Stuart M. Turnbull, 1989, Pricing and Hedging Capped Options, *Journal of Futures Markets* 9, 41–54.

Dias, José Carlos and João Pedro Nunes, 2011, Pricing Real Options under the Constant Elasticity of Variance Diffusion, *Journal of Futures Markets* 31, 230–250.

Dias, José Carlos, João Pedro Nunes, and Fernando Correia Silva, 2024, Finite Maturity Caps and Floors on Continuous Flows under the Constant Elasticity of Variance Process. Working Paper, Iscte Business School.

Hull, John C., 2018, *Options, Futures, and Other Derivatives*, 10th ed. (Pearson, New York, NY).

Keswani, Aneel and Mark B. Shackleton, 2006, How Real Option Disinvestment Flexibility Augments Project NPV, *European Journal of Operational Research* 168, 240–252.

McDonald, Robert L. and Daniel R. Siegel, 1985, Investment and the Valuation of Firms When There Is an Option to Shut Down, *International Economic Review* 26, 331–349.

McDonald, Robert L. and Mark D. Schroder, 1998, A Parity Result for American Options, *Journal of Computational Finance* 1, 5–13.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Paxson, Dean, Paulo J. Pereira, and Artur Rodrigues, 2022, Investment Timing, Capacity Choice and Optimal Floors and Ceilings, *Journal of Economic Dynamics and Control* 139, 104430.

Rubinstein, Mark and Eric Reiner, 1991, Unscrambling the Binary Code, *Risk* 4, 75–83.

Shackleton, Mark B. and RafałWojakowski, 2007, Finite Maturity Caps and Floors on Continuous Flows, *Journal of Economic Dynamics and Control* 31, 3843–3859.